Numerical Relativity

Yu Liu[1]

[1]Department of Astronomy
Huazhong University of Science and Technology
yul@hust.edu.cn

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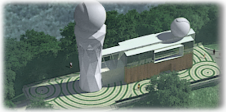
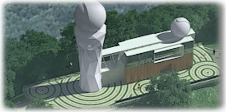




Introduction

Einstein's field equations of general relativity relate the geometry of spacetime to the local matter content in the universe according to

$$G_{ab} = 8\pi T_{ab}\,, \tag{1.1}$$

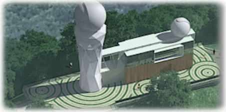
where G_{ab} is the symmetric Einstein tensor defined by

$$G_{ab} = {}^{(4)}R_{ab} - \frac{1}{2} g_{ab}\, {}^{(4)}R. \tag{1.2}$$

For all but the simplest systems, **analytic solutions for the evolution of such systems do not exist**. Hence the task of **solving Einstein's equations must be performed numerically on a computer**.

To construct algorithms to do this we first have to recast Einstein's four-dimensional field equations into a form that is suitable for numerical integration.

Cauchy Problem



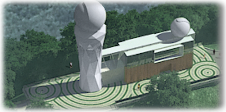
The problem of evolving the gravitational field in general relativity can be posed in terms of a traditional initial value problem or "Cauchy" problem.

> Given adequate initial (and boundary) conditions, the fundamental equations must predict the future (or past) evolution of the system.

Einstein's equation is a coupled system of nonlinear second-order partial differential equations for the metric components g_{ab}. We need specify two initial conditions g_{ab} and $\partial_t g_{ab}$.

$$\begin{array}{ccccc} t & & t+\delta t & & \ldots \\ g_{ab} & \xrightarrow{\partial_t g_{ab}} & g'_{ab} & \longrightarrow & \ldots \\ \partial_t g_{ab} & \xrightarrow{\partial_t^2 g_{ab}} & \partial_t g'_{ab} & \longrightarrow & \ldots \end{array} \qquad g_{ab} = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix}$$

Bianchi identities



The Bianchi identities

$$\nabla_b G^{ab} = 0 \tag{1.3}$$

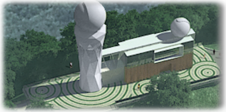
give

$$\partial_t G^{a0} = -\partial_i G^{ai} - G^{bc}\Gamma^a{}_{bc} - G^{ab}\Gamma^c{}_{bc} \tag{1.4}$$

Since no term on the right hand side of equation contains third time derivatives or higher, the four quantities G^{a0} cannot contain second time derivatives. Hence the four equations

$$G^{a0} = 8\pi T^{a0} \tag{1.5}$$

do not furnish any of the information required for the dynamical evolution of the fields.



Gauge freedom

Since g_{ab} is a tensor, it obeys the following relation for the coordinate transformation $x^{\mu} \to x'^{\mu}$:

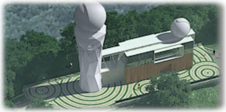
$$g_{\alpha'\beta'} = \frac{\partial x^{\alpha}}{\partial x'^{\alpha'}} \frac{\partial x^{\beta}}{\partial x'^{\beta'}} g_{\alpha\beta} \tag{1.6}$$

This shows that by an appropriate choice of the coordinates, we set at least 4 components of $g_{\alpha'\beta'}$ to be a desired form, e.g., $g_{tt} = -1$ and $g_{tk} = 0$.

> This is called the gauge freedom and for solving Einstein's equation, we have to specify a gauge condition.

This is the reason for the mismatch between the required number (10) of second time derivatives $\partial_t^2 g_{ab}$ and the available number (6) of dynamical field equations.

Einstein's equation



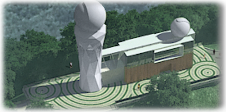
$$G^{a0} = 8\pi T^{a0} \tag{1.7}$$

supply four constraints on the initial data, i.e. four relations between g_{ab} and $\partial_t g_{ab}$ on the initial hypersurface. The only truly dynamical equations must be provided by the six remaining relations

$$G^{ij} = 8\pi T^{ij} \tag{1.8}$$

The full system of Einstein's equation is still well posed, because of the Bianchi identities (1.3). If the four initial-value equations are satisfied on some spacelike hypersurface, then the Bianchi identities guarantee that the evolution equations preserve the constraints on neighboring spacelike hypersurfaces.

Analogy Maxwell's equations



Maxwell's equations naturally split into two groups. The first group can be written as

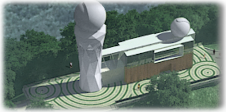
$$\begin{aligned} \mathcal{C}_E &\equiv D_i E^i - 4\pi\rho = 0 \\ \mathcal{C}_B &\equiv D_i B^i = 0 \end{aligned} \tag{1.9}$$

The above equations involve only spatial derivatives of the electric and magnetic fields and hold at each instant of time. They therefore constrain any possible configurations of the fields, and are correspondingly called **the constraint equations.**
The second group of Maxwell equations is

$$\begin{aligned} \partial_t E_i &= \epsilon_{ijk} D^j B^k - 4\pi j_i \\ \partial_t B_i &= -\epsilon_{ijk} D^j E^k \end{aligned} \tag{1.10}$$

These equations describe how the fields evolve forward in time, and are therefore called **the evolution equations.**

Formulation



The seminal work of Yvonne Choquet-Bruhat published in 1952 demonstrates that it is possible to formulate Einstein's equations as an initial value problem.

Formulation

A choice of variables and of evolution equations for them is called a formulation of the Einstein equations.

The 3+1 formalism is the most commonly used in numerical relativity, but it is certainly not the only one. It provides a nice geometric interpretation of the "foliation" of spacetime, i.e., the way in which successive time slices are chosen to fill spacetime.

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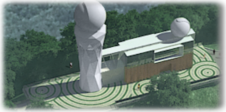
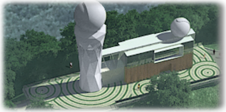
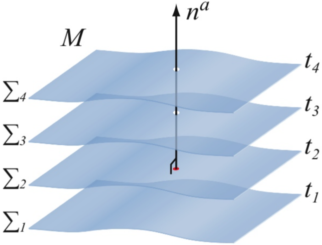




Foliations of Spacetime

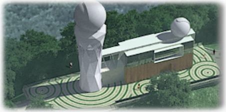
Any globally hyperbolic spacetime $(\mathcal{M}, g)$ can be foliated by a family of spacelike hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$.



By foliation or slicing, it is meant that there exists a smooth scalar field $\hat{t}$ on $\mathcal{M}$, which is regular (in the sense that its gradient never vanishes), such that each hypersurface is a level surface of this scalar field:

$$\forall t \in \mathbb{R}, \quad \Sigma_t := \{p \in \mathcal{M}, \hat{t}(p) = t\}. \tag{2.1}$$

Foliations of Spacetime



Since $\hat{t}$ is regular, the hypersurfaces Σ_t are non-intersecting:

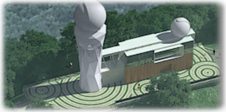
$$\Sigma_t \cap \Sigma_{t'} = \emptyset \text{ for } t \neq t'. \tag{2.2}$$

Each hypersurface Σ_t is called a leaf or a slice of the foliation. We assume that all Σ_t's are spacelike and that the foliation covers $\mathcal{M}$:

$$\mathcal{M} = \bigcup_{t \in \mathbb{R}} \Sigma_t. \tag{2.3}$$

The parameter t an then be considered as a “universal time function” (but one must be careful, t does not necessarily coincides with the proper time of any observer).

Normal vector



From t we can define the 1-form

$$\Omega_a = \nabla_a t. \tag{2.4}$$

The 4-metric g_{ab} allows us to compute the norm of Ω_a, which we call $-\alpha^{-2}$

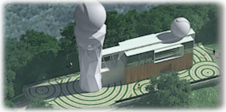
$$\|\Omega\|^2 = g^{ab}\nabla_a t \nabla_b t \equiv -\frac{1}{\alpha^2}. \tag{2.5}$$

We assume that $\alpha > 0$, so that Ω_a is timelike and the hypersurface Σ is spacelike everywhere.
We can now define the unit normal to the slices as

$$n^a \equiv -g^{ab}(\alpha\Omega_a). \tag{2.6}$$

Here the negative sign has been chosen so that n^a points in the direction of increasing t.

Spatial metric



With the normal vector we can now construct the spatial metric γ_{ab} that is induced by g_{ab} on the three-dimensional hypersurfaces Σ

$$\gamma_{ab} = g_{ab} + n_a n_b. \tag{2.7}$$

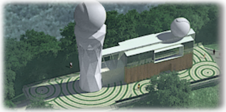
This metric allows us to compute distances within a slice Σ. To see that γ_{ab} is purely spatial, i.e., resides entirely in Σ with no piece along n^a, we contract it with the normal n^a,

$$n^a \gamma_{ab} = n^a g_{ab} + n^a n_a n_b = n_b - n_b = 0 \tag{2.8}$$

The inverse spatial metric can be found by raising the indices of γ_{ab} with g^{ab},

$$\gamma^{ab} = g^{ac} g^{bd} \gamma_{cd} = g^{ab} + n^a n^b. \tag{2.9}$$

Projection operators



Next we break up 4-dimensional tensors by decomposing them into a purely spatial part, which lies in the hypersurfaces Σ, and a timelike part, which is normal to the spatial surface. To do so, we need two projection operators.

The first one, which projects a 4-dimensional tensor into a spatial slice, can be found by raising only one index of the spatial metric γ_{ab}

$$\gamma^a{}_b \equiv \delta^a{}_b + n^a n_b. \tag{2.10}$$

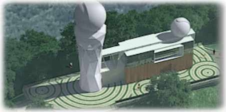
Similarly, we may define the normal projection operator as

$$N^a{}_b \equiv -n^a n_b. \tag{2.11}$$

We can now use these two projection operators to decompose any tensor into its spatial and timelike parts. For example, we can write an arbitrary vector v^a as

$$v^a = \delta^a{}_b v^b = (\gamma^a{}_b + N^a{}_b)\, v^b. \tag{2.12}$$

Field equations in the 3+1 formalism



In order to write down the Einstein equations in 3+1 form, we will use the projection operator $\gamma^a{}_b$, together with the normal vector n^a, to separate Einstein's equations in three groups:

- Normal projection (1 equation):

$$n^a n^b \left(G_{ab} - 8\pi T_{ab}\right) = 0 \tag{2.13}$$

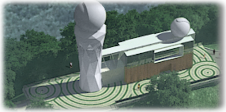
- Mixed projections (3 equations):

$$\gamma^b{}_a n^c \left(G_{bc} - 8\pi T_{bc}\right) = 0 \tag{2.14}$$

- Projection onto the hypersurface (6 equations):

$$\gamma^c{}_a \gamma^d{}_b \left(G_{cd} - 8\pi T_{cd}\right) = 0 \tag{2.15}$$

Three-dimensional covariant derivative



We can construct three-dimensional covariant derivative by projecting all indices present in a 4-dimensional covariant derivative into Σ.
For a scalar f, for example, we define

$$D_a f \equiv \gamma^b{}_a \nabla_b f, \tag{2.16}$$

The three-dimensional Riemann tensor associated with γ_{ab} is defined by requiring that

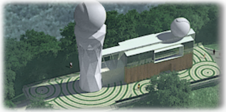
$$\begin{aligned} R^d{}_{cba} w_d &= 2D_{[a}D_{b]} w_c, \\ R^d{}_{cba} n_d &= 0, \end{aligned} \tag{2.17}$$

for any spatial vector w_d.

Note

The three-dimensional covariant derivative is associated with the spatial metric γ_{ab}, that is,

$$D_a \gamma_{bc} = 0 \tag{2.18}$$

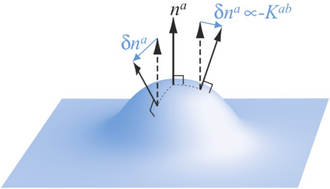


Extrinsic curvature

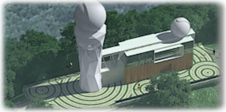
Using the projection operator, the extrinsic curvature tensor is defined as:

$$K_{ab} \equiv -\gamma^c{}_a \gamma^d{}_b \nabla_c n_d. \tag{2.19}$$

By definition, the extrinsic curvature is symmetric and purely spatial. The extrinsic curvature therefore provides information on how much the normal vector changes from point to point across a spatial hypersurface.



Extrinsic curvature



Alternatively, we can write the extrinsic curvature as

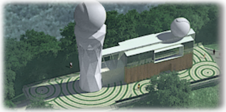
$$K_{ab} = -\frac{1}{2}\mathcal{L}_{\mathbf{n}}\gamma_{ab}, \tag{2.20}$$

where $\mathcal{L}_{\mathbf{n}}$ denotes the Lie derivative along n^a. Since n^a is a timelike vector, the extrinsic curvature can be interpretation as the “time derivative” of the spatial metric γ_{ab} as seen by the Eulerian observers.

Note

We already have an evolution equation for the spatial metric γ_{ij}. In order to close the system we still need an evolution equation for K_{ij}. It is important to notice that until now we have only worked with purely geometric concepts, and we have not used the field equations at all. **It is precisely from the field equations that we will obtain the evolution equations for K_{ij}.**

Hamiltonian constraint



From the normal projection we find the following equation:

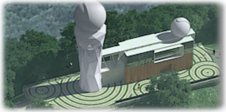
$$R + K^2 - K_{ab}K^{ab} = 16\pi\rho, \tag{2.21}$$

where R is the Ricci scalar associated with the 3-metric, K is the trace of the extrinsic curvature tensor, and ρ is the energy density of matter as measured by the Eulerian observers:

$$\rho \equiv n_a n_b T^{ab}. \tag{2.22}$$

Equation (2.21) contains no time derivatives. Because of this, the equation is not a dynamical equation but rather a "constraint" of the system. As it is related with the energy density, it is known as the "hamiltonian" constraint.

Momentum constraints



From the mixed projection of the field equations we find:

$$D_b K^b{}_a - D_a K = 8\pi S_a, \tag{2.23}$$

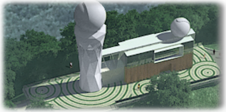
where now S_a is the momentum flux of matter as measured by the Eulerian observers:

$$S_a \equiv -\gamma^b{}_a n^c T_{bc}. \tag{2.24}$$

Equation (2.23) again has no time derivatives, so it is another constraint. These equations are known as the "momentum" constraints.

The existence of the constraints implies that in general relativity it is not possible to specify arbitrarily all 12 dynamical quantities $\{\gamma_{ab}, K_{ab}\}$ as initial conditions. The initial data must already satisfy the constraints, otherwise we will not be solving Einstein's equations.

Evolution equations



The remaining 6 equations are obtained from the projection onto the hypersurface and contain the true dynamics of the system.

$$
\begin{aligned}
\mathcal{L}_{\mathrm{n}} K_{ab} = & -\frac{1}{\alpha} D_a D_b \alpha + \left(R_{ab} - 2K_{ac}K^{c}{}_{b} + KK_{ab}\right) \\
& - 8\pi \left(S_{ab} - \frac{1}{2}\gamma_{ab}(S - \rho)\right),
\end{aligned}
\tag{2.25}
$$

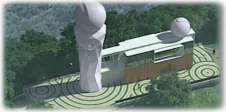
where S_{ab} is the stress tensor of matter, defined as:

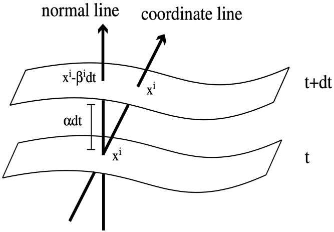
$$
\begin{aligned}
S_{ab} &\equiv \gamma^{c}{}_{a}\gamma^{d}{}_{b} T_{cd}, \\
S &\equiv S^{a}{}_{a}.
\end{aligned}
\tag{2.26}
$$

Equations (2.20) and (2.25) form a closed system of evolution equations.

The Lie derivative along n^a, $\mathcal{L}_{\mathbf{n}}$, is not a natural time derivative.

Two neighboring spatial hypersurfaces





Let t^a be a timelike vector field on the spacetime which is the tangent to the time axis, $t^a = (\partial/\partial t)^a$. t^a will connect points with the same spatial coordinates on neighboring time slices.

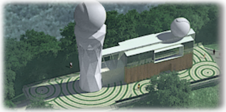
Note that t^a is not always normal to Σ_t, and thus, it has components both on Σ_t and along n^a. We decompose t^a into two components as

$$\alpha := -t^a n_a, \quad \beta^b := t^a \gamma^b{}_a \tag{2.27}$$

Here α and β^a are called the lapse function and the shift vector, respectively. Using these quantities, t^a is written as

$$t^a = \alpha n^a + \beta^a \tag{2.28}$$

Coordinate congruence t^a



Consider now the Lie derivative of γ_{ab} and K_{ab} along t^a. We find

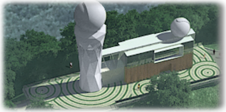
$$\begin{aligned}\mathcal{L}_{\mathrm{t}} K_{ab} = & -D_a D_b \alpha + \alpha \left(R_{ab} - 2K_{ac}K^c{}_b + KK_{ab}\right) \\ & - 8\pi\alpha \left(S_{ab} - \frac{1}{2}\gamma_{ab}(S-\rho)\right) + \mathcal{L}_\beta K_{ab}\end{aligned} \tag{2.29}$$

and

$$\mathcal{L}_{\mathbf{t}}\gamma_{ab} = -2\alpha K_{ab} + \mathcal{L}_\beta \gamma_{ab} \tag{2.30}$$

The lapse and shift do not appear in the constraint equations. They appear in the evolution equations, but without time derivatives. These quantities may be chosen freely, without changing the physical solution of the Einstein equation. However, the lapse and shift do not completely specify the coordinate gauge. They specify **our temporal gauge choice**.

Choosing Basis Vectors

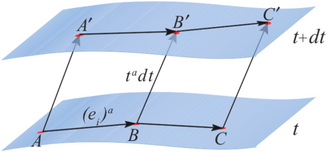


So far, we have expressed our equations in a covariant, coordinate independent manner, i.e. the basis vectors e^a have been completely arbitrary. It is quite intuitive that things will simplify if we adopt a coordinate system that reflects our 3 + 1 split of spacetime in a natural way.
To do so, we first introduce a basis of three spatial vectors $e^a_{(i)}$ that reside in a particular time slice Σ

$$n_a e^a_{(i)} = 0. \tag{2.31}$$

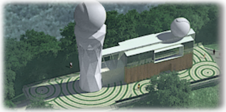
We extend our spatial vectors to other slices Σ by Lie dragging along t^a,

$$\mathcal{L}_{\mathbf{t}} e^a_{(i)} = 0, \tag{2.32}$$



As the fourth basis vector we pick $e^a_{(0)} = t^a$, the Lie derivative along t^a reduces to a partial derivative with respect to t: $\mathcal{L}_{\mathrm{t}} = \partial_t$.

Tensor components



Since the $e^a_{(i)}$ span Σ, equation (2.31) implies that the covariant spatial components of the normal vector have to vanish,

$$n_i = 0. \tag{2.33}$$

Since spatial tensors vanish when contracted with the normal vector, this also means that all components of spatial tensors with a contravariant index equal to zero must vanish. For the shift vector, for example, this implies $n_a\beta^a = n_0\beta^0 = 0$ and hence

$$\beta^a = \left(0, \beta^i\right). \tag{2.34}$$

By definition, $t^a = (1, 0, 0, 0)$. Solving equation (2.28) for n^a then yields the contravariant components

$$n^a = \left(\alpha^{-1}, -\alpha^{-1}\beta^i\right), \tag{2.35}$$

and from the normalization condition $n_a n^a = -1$ we find

$$n_a = (-\alpha, 0, 0, 0). \tag{2.36}$$

From the definition of the spatial metric (2.7) we have

$$\gamma_{ij} = g_{ij}, \tag{2.37}$$

meaning that the metric on Σ is just the spatial part of the four-metric.

Since zeroth components of spatial contravariant tensors have to vanish, we also have $\gamma^{a0} = 0$. The inverse metric can therefore be expressed as

$$g^{ab} = \gamma^{ab} - n^a n^b = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta^i \\ \alpha^{-2}\beta^j & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{pmatrix}. \tag{2.38}$$

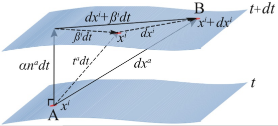
We can now invert (2.38) and find the components of the four-dimensional metric

$$g_{ab} = \begin{pmatrix} -\alpha^2 + \beta_l\beta^l & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix}. \tag{2.39}$$

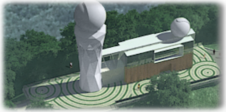
Equivalently, the line element may be decomposed as

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right). \tag{2.40}$$

which is often refered to as the metric in 3 + 1 form.



ADM Equations



The entire content of any spatial tensor is available from their spatial components. This is obviously true for contravariant components, since their zeroth component vanishes, but also holds covariant components. Therefore, the entire content of the decomposed Einstein equations is contained in their spatial components alone.

We can rewrite the Hamiltonian constraint (2.21),

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho, \tag{2.41}$$

the momentum constraint (2.23),

$$D_j\left(K^{ij} - \gamma^{ij}K\right) = 8\pi S^i, \tag{2.42}$$

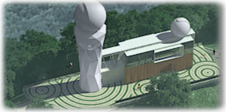
the evolution equation for the spatial metric (2.20),

$$\partial_t\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i, \tag{2.43}$$

and the evolution equation for the extrinsic curvature (2.25),

$$\begin{aligned}\partial_t K_{ij} = &- D_iD_j\alpha + \alpha\left(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij}\right) - 8\pi\alpha\left(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)\right)\\ &+ \beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k.\end{aligned} \tag{2.44}$$

Gauge



The freedom in choosing the gauge variables is a mixed blessing. On the one hand, it allows us to choose things in a way that simplifies the equations, or makes the solution better behaved. On the other hand, we are immediately faced with the question: **What is a "good" choice for the functions α and β^i?** Remember that this decision has to be made even before we start the evolution.

Let us first consider the simplest possible choice,

$$\alpha = 1, \quad \beta^i = 0. \tag{2.45}$$

In the context of numerical relativity this gauge choice is often called geodesic slicing.

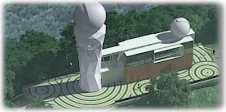
Coordinate observers move with 4-velocities $u^a = t^a$. Thus with $\beta^i = 0$, coordinate observers coincide with normal observers $u^a = n^a$. With $\alpha = 1$, the proper time intervals that they measure agree with coordinate time intervals. Their acceleration is

$$a_b = n^b \nabla_b n_a = D_b \ln \alpha = 0. \tag{2.46}$$

Evidently, since their acceleration vanishes, normal observers are freely-falling and therefore follow geodesics.

Despite its simplicity, geodesic slicing tends to form coordinate singularities very quickly during an evolution. This result is not surprising, since geodesics tend to focus in the presence of gravitating sources.

Analogy Maxwell's equations



It is possible to bring Maxwell's equations into a form that is closer to the 3+1 form of Einstein's equations. To do so, we introduce the vector potential $A^a = \left(\Phi, A^i\right)$ and write B^i as

$$B_i = \epsilon_{ijk} D^j A^k \tag{2.47}$$

By construction, B_i automatically satisfies the constraint $D_i B^i = 0$. The two evolution equations can be rewritten in terms of E_i and A_i

$$\begin{aligned} \partial_t A_i &= -E_i - D_i \Phi \\ \partial_t E_i &= D_i D^j A_j - D^j D_j A_i - 4\pi j_i \end{aligned} \tag{2.48}$$

With the vector potential A_i we have introduced **a gauge freedom into electrodynamics which is expressed in the freely specifiable gauge variable Φ.**

It is instructive to compare the standard 3 + 1 gravitational field equations with Maxwell's equations of electrodynamics.

$$\begin{aligned} \partial_t A_i &= -E_i - D_i\Phi \\ \partial_t E_i &= D_i D^j A_j - D^j D_j A_i - 4\pi j_i \end{aligned}$$

$$\begin{aligned} \partial_t \gamma_{ij} =& -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \\ \partial_t K_{ij} =& -D_i D_j \alpha + \alpha \left(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij}\right) \\ & - 8\pi\alpha \left(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)\right) \\ & + \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{kj} D_i \beta^k . \end{aligned}$$

If we identify the vector potential A_i with the spatial metric γ_{ij} and the electric field E_i with the extrinsic curvature K_{ij}. The right-hand sides of equation $\partial_t\gamma_{ij}$ involve a field variable and a spatial derivative of a gauge variable, while the right-hand sides of equation $\partial_t K_{ij}$ involve matter sources as well as second spatial derivatives of the field variable. **The most important differences between the two theories are also obvious: electromagnetism is a linear, vector field theory, while general relativity is a nonlinear, tensor field theory.**

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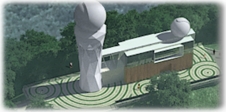
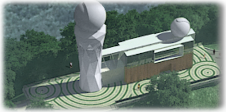


Hyperbolic



Consider a system of evolution equations:

$$\partial_t \mathbf{u} + \mathbf{A}^i \cdot \partial_i \mathbf{u} = \mathbf{S}. \tag{3.1}$$

where $\mathbf{u}$ is a solution vector, $\mathbf{S}$ is a source vector, and where we have called the matrix $\mathbf{A}$ the velocity matrix.

Most differential evolution equations in physics can be written in this form. In the case when there are higher order derivatives, one can always define auxiliary variables in order to obtain a first order system.

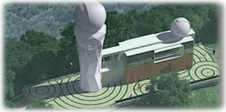
We call a problem well-posed if we can define some norm $\| \ldots \|$ so that the norm of the solution vector satisfies

$$\left\| \mathbf{u}\left(t, x^i\right) \right\| \leq k e^{\alpha t} \left\| \mathbf{u}\left(0, x^i\right) \right\| \tag{3.2}$$

for all times $t \geq 0$. Here k and α are two constants that are independent of the initial data $\mathbf{u}\left(0, x^i\right)$. Stated differently, solutions of a well-posed problem cannot increase more rapidly than exponentially.

From a numerical perspective, however, exponentially growing modes are still very bad, and can easily terminate a simulation after only a short time. From a numerical perspective, then, **well-posedness is a necessary but not a sufficient condition**.

Hyperbolicity properties of the ADM



Even thought the ADM equations are the starting point of numerical relativity, in practice they have not turned out to be very well behaved with respect to small constraint violations. **The ADM equations are not very stable with respect to constraint violations**. This has prompted much effort in developing alternative formulations of the 3+1 Einstein equations.

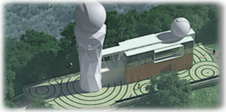
$$R_{ij} = \frac{1}{2}\gamma^{kl}\left(\underbrace{\partial_i\partial_l\gamma_{kj} + \partial_k\partial_j\gamma_{il} - \partial_i\partial_j\gamma_{kl}}_{\text{mixed derivative terms}} - \partial_k\partial_l\gamma_{ij}\right) + \gamma^{kl}\left(\Gamma^m{}_{il}\Gamma_{mkj} - \Gamma^m{}_{ij}\Gamma_{mkl}\right)$$

Analogy Maxwell's Equations

In electrodynamics, it would be desirable to eliminate the mixed derivative terms $D_iD^jA_j$. The most straightforward approach is to choose a Coulomb gauge $D^iA_i = 0$.

In general relativity, an analogous approach can be taken by choosing harmonic coordinates.

Harmonic Coordinates



Consider a contraction of the four dimensional connection coefficients

$$
{}^{(4)}\Gamma^{a} \equiv g^{bc}\,{}^{(4)}\Gamma^{a}{}_{bc} = -\frac{1}{|g|^{1/2}}\partial_b\left(|g|^{1/2}g^{ab}\right) = 0 \tag{3.3}
$$

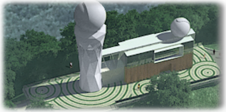
This bring the four-dimensional Ricci tensor ${}^{(4)}R_{ab}$ into a particularly simple form.

Inserting the metric (2.38) into equation (3.3) shows that in harmonic coordinates the lapse and shift satisfy the coupled set of hyperbolic equations

$$
\begin{aligned}
\left(\partial_t - \beta^j\partial_j\right)\alpha &= -\alpha^2 K \\
\left(\partial_t - \beta^j\partial_j\right)\beta^i &= -\alpha^2\left(\gamma^{ij}\partial_j \ln\alpha + \gamma^{jk}\Gamma^{i}{}_{jk}\right)
\end{aligned} \tag{3.4}
$$

This approach has disadvantages, too. In general relativity, these coordinates may lead to coordinate singularities.

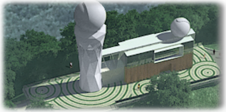
BSSN Formulation



The BSSN scheme has been devised by Shibata and Nakamura in 1995 [Shibata and Nakamura, 1995]. It has been re-analyzed by Baumgarte and Shapiro in 1999 [Baumgarte and Shapiro, 1998], with a slight modification, and bears since then the name BSSN for Baumgarte-Shapiro-Shibata-Nakamura.
The BSSN formulation is obtained from the ADM formulation by introducing the new variables. The BSSN formalism adopts a similar strategy to simplify the three-dimensional, spatial Ricci tensor by absorb the mixed second derivatives.

- Conformal Transformations
- Conformal connection function
- Constraint propagation and damping

Conformal Transformations



First of all the metric is split into a conformal metric

$$\bar{\gamma}_{ij} = e^{-4\phi}\gamma_{ij} \tag{3.5}$$

and a conformal factor

$$\phi = \frac{1}{12}\log\gamma \tag{3.6}$$

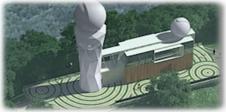
The BSSN formulation also assumes that $\bar{\gamma} = \det(\bar{\gamma}_{ij}) = 1$ in Cartesian coordinates. Loosely speaking, the conformal factor absorbs the overall scale of the metric, which leaves five degrees of freedom in the conformally related metric.
We can split the Ricci tensor into two terms

$$R_{ij} = \bar{R}_{ij} + R_{ij}^{\phi} \tag{3.7}$$

where R_{ij}^{ϕ} depends only on the conformal function ϕ.
Then the extrinsic curvature is split into its trace K and a traceless part:

$$\bar{A}_{ij} = e^{-4\phi}\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right) \tag{3.8}$$

Conformal connection function



We can now define "conformal connection functions"

$$\bar{\Gamma}^i \equiv \bar{\gamma}^{jk}\bar{\Gamma}^i{}_{jk} = -\partial_j \bar{\gamma}^{ij}. \tag{3.9}$$

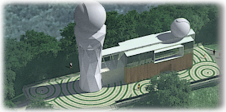
Here the $\bar{\Gamma}^i{}_{jk}$ are the connection coefficients associated with $\bar{\gamma}_{ij}$. In terms of these conformal connection functions we can now write the Ricci tensor as

$$\bar{R}_{ij} = -\frac{1}{2}\bar{\gamma}^{lm}\partial_m\partial_l\bar{\gamma}_{ij} + \bar{\gamma}_{k(i}\partial_{j)}\bar{\Gamma}^k + \bar{\Gamma}^k\bar{\Gamma}_{(ij)k} + \bar{\gamma}^{lm}\left(2\bar{\Gamma}^k{}_{l(i}\bar{\Gamma}_{j)km} + \bar{\Gamma}^k{}_{im}\bar{\Gamma}_{klj}\right). \tag{3.10}$$

Since the $\bar{\Gamma}^i$ are evolved as independent functions, the defining relation (3.9) serves as a new constraint equation. Adopting this approach requires us to derive separate evolution equations for the $\bar{\Gamma}^i$. We interchange a partial time and space derivative in the definition (3.9) to obtain

$$\begin{aligned}\partial_t\bar{\Gamma}^i = &-2\tilde{A}^{ij}\partial_j\alpha + 2\alpha\left(\bar{\Gamma}^i_{jk}\tilde{A}^{kj} - \frac{2}{3}\bar{\gamma}^{ij}\partial_j K - 8\pi\bar{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\partial_j\phi\right)\\ &+ \beta^j\partial_j\bar{\Gamma}^i - \bar{\Gamma}^j\partial_j\beta^i + \frac{2}{3}\bar{\Gamma}^i\partial_j\beta^j + \frac{1}{3}\bar{\gamma}^{li}\partial_l\partial_j\beta^j + \bar{\gamma}^{lj}\partial_j\partial_l\beta^i\end{aligned} \tag{3.11}$$

The BSSN equations



We call ϕ, K, $\bar{\gamma}_{ij}$, $\tilde{A}_{ij}$ and $\bar{\Gamma}^i$ the BSSN variables. In terms of these variables the Hamiltonian constraint becomes

$$0 = \mathcal{H} = \bar{\gamma}^{ij}\bar{D}_i\bar{D}_j e^{\phi} - \frac{e^{\phi}}{8}\bar{R} + \frac{e^{5\phi}}{8}\tilde{A}_{ij}\tilde{A}^{ij} - \frac{e^{5\phi}}{12}K^2 + 2\pi e^{5\phi}\rho \tag{3.12}$$

while the momentum constraint becomes

$$0 = \mathcal{M}^i = \bar{D}_j\left(e^{6\phi}\tilde{A}^{ji}\right) - \frac{2}{3}e^{6\phi}\bar{D}^i K - 8\pi e^{6\phi}S^i \tag{3.13}$$

The evolution equation for γ_{ij} splits into two equations,

$$\begin{aligned} \partial_t\phi &= -\tfrac{1}{6}\alpha K + \beta^i\partial_i\phi + \tfrac{1}{6}\partial_i\beta^i \\ \partial_t\bar{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij} + \beta^k\partial_k\bar{\gamma}_{ij} + \bar{\gamma}_{ik}\partial_j\beta^k + \bar{\gamma}_{kj}\partial_i\beta^k - \tfrac{2}{3}\bar{\gamma}_{ij}\partial_k\beta^k \end{aligned} \tag{3.14}$$

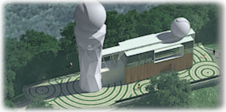
while the evolution equation for K_{ij} splits into the two equations

$$\begin{aligned} \partial_t K =& -\gamma^{ij}D_jD_i\alpha + \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) + 4\pi\alpha(\rho + S) + \beta^i\partial_i K \\ \partial_t\tilde{A}_{ij} =& e^{-4\phi}\left(-\left(D_iD_j\alpha\right)^{TF} + \alpha\left(R_{ij}^{TF} - 8\pi S_{ij}^{TF}\right)\right) + \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}_j^l\right) \\ & + \beta^k\partial_k\tilde{A}_{ij} + \tilde{A}_{ik}\partial_j\beta^k + \tilde{A}_{kj}\partial_i\beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k\beta^k \end{aligned} \tag{3.15}$$

The $\bar{\Gamma}^i$ are now treated as independent functions that satisfy their own evolution equations,

$$\begin{aligned} \partial_t\bar{\Gamma}^i =& -2\tilde{A}^{ij}\partial_j\alpha + 2\alpha\left(\bar{\Gamma}^i_{jk}\tilde{A}^{kj} - \frac{2}{3}\bar{\gamma}^{ij}\partial_j K - 8\pi\bar{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\partial_j\phi\right) \\ & + \beta^j\partial_j\bar{\Gamma}^i - \bar{\Gamma}^j\partial_j\beta^i + \frac{2}{3}\bar{\Gamma}^i\partial_j\beta^j + \frac{1}{3}\bar{\gamma}^{li}\partial_l\partial_j\beta^j + \bar{\gamma}^{lj}\partial_j\partial_l\beta^i \end{aligned} \tag{3.16}$$

Constraint propagation and damping



The stability properties of the basic BSSN system itself can be enhanced further by explicitly adding Hamiltonian and/or momentum constraints.

Analogy Maxwell's Equation

To illustrate this effect in a simple setting, let us return to Maxwell's equations

$$\mathcal{C} \equiv D^i E_i - 4\pi \rho_e. \tag{3.17}$$

By differentiating the continuity equation $D_i j^i + \partial_t \rho_e = 0$ and using the evolution equation of E_i, it is easy to show that the time derivative of this constraint variable vanishes identically,

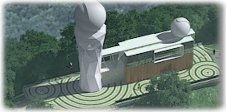
$$\begin{aligned} \partial_t \mathcal{C} &= \partial_t \left(D^i E_i - 4\pi \rho_e \right) = D^i \partial_t E_i - 4\pi \partial_t \rho_e \\ &= -D^i D^j D_j A_i + D^i D_i D^j A_j - 4\pi \left(D^i j_i + \partial_t \rho_e \right) = 0 \end{aligned} \tag{3.18}$$

This indicates that any violation of the constraint ($\mathcal{C} \neq 0$) will persist and not propagate away.
Now consider adding the constraint violation parameter C times some constant a^2 in the evolution equation. Then constraint violations measured by the parameter C satisfy the wave equation

$$\left(-\partial_t^2 + a^2 D_i D^i \right) \mathcal{C} = 0 \tag{3.19}$$

If the condition $\mathcal{C} = \partial_t \mathcal{C} = 0$ holds initially, then the two systems are equivalent analytically. However, the two systems behave very differently numerically, since any numerical (e.g. roundoff) error will lead to a constraint violation $|\mathcal{C}| > 0$, which will then evolve differently in the two systems.

Gauge conditions



There are several points to be kept in mind for the choice of the gauge condition in numerical relativity.

- Avoiding the appearance of coordinate singularities
- Avoiding black hole singularities

The most widely adopted formulation for compact binary evolutions is the highly robust "BSSN/MP" formulation, which combines the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) 3+1 decomposition of Einstein's equations with the "moving puncture" (MP) gauge conditions implement the 1+log/Γ-freezing gauge evolution equations.

- Slicing:

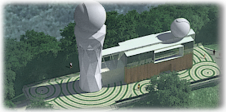
$$\alpha = 1 + \log\gamma \tag{3.20}$$

- Shift:

$$\begin{aligned} \partial_t\beta^i &= \tfrac{3}{4}\alpha B^i \\ \partial_t B^i &= \partial_t\tilde{\Gamma}^i - \eta B^i \end{aligned} \tag{3.21}$$

The results of the numerical solution obtained with different gauge conditions will be different, but only in the "gauge-dependent" quantities. On the other hand, physical observables, such as scalar quantities or gravitational waves (when extracted in the wave zone) will not depend on the specific choice of gauge conditions, i.e., they will be "gauge-invariant".

Reference



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Shibata, M. and Nakamura, T. (1995).
Evolution of three-dimensional gravitational waves: Harmonic slicing case.
Physical Review D, 52(10):5428–5444.

Book:

- Numerical Relativity: Solving Einstein's Equations on the Computer

Courses:

- Thomas Baumgarte - Numerical relativity
- Physics Unsimplified - Lectures on General Relativity